# THE MULTIDIMENSIONAL MANHATTAN NETWORKS *

F. COMELLAS, C. DALFÓ, AND M.A. FIOL †

**Abstract.** The $n$-dimensional Manhattan network $M_n$—a special case of $n$-regular digraph—is formally defined and some of its structural properties are studied. In particular, it is shown that $M_n$ is a Cayley digraph, which can be seen as a subgroup of the $n$-dim version of the wallpaper group *pgg*. These results induce a useful new presentation of $M_n$, which can be applied to design a (shortest-path) local routing algorithm and to study some other metric properties. Also it is shown that the $n$-dim Manhattan networks are Hamiltonian and, in the standard case (that is, dimension two), they can be decomposed in two arc-disjoint Hamiltonian cycles. Finally, some results on the connectivity and distance-related parameters of $M_n$, such as the distribution of the node distances and the diameter are presented.

**Key words.** Manhattan street network, Cayley digraph, diameter, Hamiltonian cycle.

**AMS subject classifications.** 05C20, 05C25, 05C12, 05C45, 90B10

## 1. Introduction.

The study of a class of toroidal directed networks, commonly known in the literature as Manhattan (Street) Networks, has received significant attention since they were introduced independtly (in different contexts) by Morillo *et al.* [19] and Maxemchuk [17] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan (or *l'Eixample* in downtown Barcelona).

In [19] the networks were related to plane tessellations, and this association facilitates the study of some metric properties. Most of the work on Manhattan networks has been devoted to the computation of the average distance [16] and the generation of routing schemes [17] for the 2-dimensional case. These results have been usually inspired by conjectures supported by computer simulations. The study of spanning trees [6] in a Manhattan network has allowed the computation of the diameter and the design of a multi-port broadcasting algorithm. More recently, Varvarigos in [22] evaluated the mean internodal distance of such a network, and provided also a shortest path routing algorithm and two edge-disjoint Hamiltonian cycles in the 2-dimensional case $N \times N$. The multidimensional natural extension of the Manhattan networks has been considered by Banerjee *et al.*, see [1, 2], with the determination of the average distance of a 3-dimensional Manhattan network, and a conjecture for higher dimensions. Chung and Agrawal [7] studied the diameter and provided routing schemes for a 3-dimensional construction based on 2-dimensional Manhattan networks, although the proposed resulting network is not strictly a 3-dimensional Manhattan network.

In this paper we give a formal definition of an $n$-dimensional Manhattan network $M_n$, together with its main properties, and provide analytical determinations of some of its distance-related parameters, such as the diameter. As a useful result, it is shown that $M_n$ is a Cayley digraph. This fact allows us to introduce a new presentation of $M_n$, which can be applied to prove a number of results, such as the design of a (shortest-path) local routing algorithm. We also give some details on the cycle structure and a proof of the Hamiltonicity of these digraphs. Finally, some results on

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the connectivity and distance-related parameters of $M_n$, such as the distribution of the node distances and the diameter are presented.

**1.1. Some notation on digraphs.** Recall that a digraph $G = (V, A)$ consists of a set of *vertices* $V$, together with a set of *arcs* $A$, which can be seen as ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc $(u, v)$ is usually depicted as an arrow with *initial vertex* $u$ and *terminal vertex* $v$; that is, $u \to v$. The *indegree* $\delta^-(u)$ (respectively, *outdegree* $\delta^+(u)$) of a vertex $u$ is the number of arcs with initial (respectively, terminal) vertex $u$. Then $G$ is $\delta$*-regular* when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$. Given a digraph $G = (V, A)$, its *converse* digraph $\overline{G} = (V, \overline{A})$ is obtained from $G$ by reversing all the orientations of the arcs in $A$; that is, $(u, v) \in \overline{A}$ if and only if $(v, u) \in A$.

Given a group $\Gamma$ with (finite) generating set $\Delta$, the *Cayley digraph* $\mathrm{Cay}(\Gamma, \Delta)$ has vertexs representing the elements of $\Gamma$, and arcs of the form $(g, h\partial)$ where $g, h \in \Gamma$ and $\partial \in \Delta$. The Cayley digraph $\mathrm{Cay}(\Gamma, \Delta)$ is a vertex-transitive strongly connected regular digraph.

The well known Sabidussi's characterization result [20] states that a digraph is a Cayley digraph (for some pair $\Gamma$, $\Delta$) if and only if its automorphism group contains a regular subgroup.

An *homomorphism* $\Psi$ from a digraph $G$ to a digraph $H$ is a mapping from the vertex set of $G$ to the vertex set of $H$ preserving adjacencies; that is, if $(u, v)$ is an arc of $G$, then $(\Psi(u), \Psi(v))$ is an arc of $H$. Moreover, if both digraphs are arc-colored (all arcs with the same initial or terminal vertex receive different colors) and $\Psi$ preserves the colors, we say that $\Psi$ is a *colored homomorphism* or simply that it is *color-preserving*.

Other standard definitions and basic results about graphs and digraphs not recalled here can be found in [3, 5].

**2. The Multidimensional Manhattan Network.** Recall that the standard Manhattan (Street) Network $M(N_1, N_2)$ was defined as a 2-regular digraph in the following way. Every vertex is represented by a pair of integers $\boldsymbol{u} = (u_1, u_2)$, with $0 \le u_i \le N_i$, for some even integers $N_i$, $i = 1, 2$ and vertex $\boldsymbol{u}$ has two outgoing arcs: one horizontal $(u_1 \pm 1, u_2)$; and the other vertical $(u_1, u_2 \pm 1)$ (where the sign depends on the parity of the other component and arithmetic must be understood modulo $N_i$). More precisely, a horizontal arc points to *est* (respectively, *west*) when it is on an *even* (respectively, *odd*) row. Similarly, a vertical arc points to *north* (respectively, *south*) if it is on an *even* (respectively, *odd*) column.

Locally the structure is as shown in Fig. 2.1, and corresponds to a standard pattern for the allowed traffic directions in some neighborhoods of our modern cities, like New York or Barcelona, with their system of straight orthogonal streets. In most of the papers [17, 6, 8] the above mentioned toroidal version of $M_2$ was considered, whereas in [19] the aim was to construct the locally-Manhattan network with maximum number of vertices for a given diameter.

A formal definition of the toroidal version, which applies also for the $n$-dimensional case, is the following:

DEFINITION 2.1. *Given $n$ even positive integers $N_1, N_2, \ldots, N_n$, the $n$-dim Manhattan network $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an $n$-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \le u_i \le N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is*

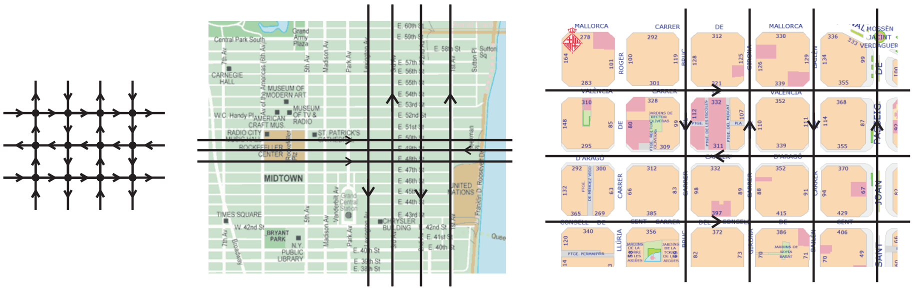


FIG. 2.1. *The local pattern of a Manhattan network and two real-life examples: Orthogonal streets of Manhattan and l'Eixample in Barcelona.*

*defined by the following adjacencies (here called $i$-*arcs*):*

$$(2.1)\quad (u_1,\dots,u_i,\dots,u_n) \quad\rightarrow\quad (u_1,\dots,u_i+(-1)^{\sum_{j\neq i}u_j},\dots,u_n) \qquad (1\le i\le n).$$

Therefore, $M_n$ is a $n$-regular digraph on $N=\prod_{i=1}^n N_i$ vertices.

In particular notice that, when $N_i=2$, $1\le i\le n$, we always have $(-1)^{\sum_{j\neq i}u_j}=1$. Hence, in this case the $n$-dimensional Manhattan network is isomorphic to the symmetric digraph $Q_n^*$, with $Q_n$ being the hypercube of dimension $n$ or $n$-*cube*.

Some other simple consequences of the definition of $M_n$ are presented in the following lemma:

LEMMA 2.2. *Every $n$-dimensional Manhattan network $M_n=M(N_1,N_2,\dots,N_n)$ satisfy the following properties:*

- $(a)$ *Given any permutation $\sigma$ of the numbers $N_1,N_2,\dots,N_n$, say $P_1,P_2,\dots,P_n$, the Manhattan networks $M_n$ and $M_n^\sigma=M(P_1,P_2,\dots,P_n)$ are isomorphic digraphs.*
- $(b)$ *$M_n$ is isomorphic to its converse $\overline{M}_n$.*
- $(c)$ *For any $n-k$ fixed integers $x_i\in\mathbb{Z}_{N_i}$, $i=k+1,k+2,\dots,n$, the subdigraph of $M_n$ induced by the vertices of the form $(u_1,u_2,\dots,u_k,x_{k+1},\dots,x_n)$ is either the $k$-dim Manhattan network $M_k=M(N_1,N_2,\dots,N_k)$ or its converse $\overline{M}_k$, depending on whether $\alpha:=\sum_{i=k+1}^n x_n$ is even or odd, respectively.*
- $(d)$ *$M_n$ is both a $2^n$-partite and bipartite digraph.*
- $(e)$ *There exists an homomorphism from $M_n$ to the symmetric digraph of the hypercube $Q_n^*$.*

*Proof.* The result in $(a)$ is clear with $\sigma$ acting on the (components of the) vertices of $M_n$. To prove $(b)$, note that in the converse digraph $\overline{M}_n$, the adjacencies are just

$$(2.2)\quad (u_1,\dots,u_i,\dots,u_n) \quad\rightarrow\quad (u_1,\dots,u_i-(-1)^{\sum_{j\neq i}u_j},\dots,u_n) \qquad (1\le i\le n).$$

Hence, it is readily checked that the mapping $\varphi: V(M_n)\rightarrow V(\overline{M}_n)$ defined by $\varphi(\boldsymbol{u})=-\boldsymbol{u}$ is the required isomorphism. The result in $(c)$ follows from the "converse adjacencies" in (2.2) and the fact that $(-1)^{\sum_{j=1,j\neq i}^k u_i+\alpha}=\pm(-1)^{\sum_{j=1,j\neq i}^k u_i}$ depending on the parity of $\alpha$. Moreover, $(d)$ holds since $M_n$ is a $2^n$-partite digraph with independent sets $V_{\boldsymbol{b}}$, where $\boldsymbol{b}=(b_1,b_2,\dots,b_n)$ is an $n$-binary string. A vertex $\boldsymbol{u}=(u_1,u_2,...,u_n)$ belongs to $V_{\boldsymbol{b}}$ when the parities of $u_i$ and $b_i$ coincide for every $1\le i\le n$. In particular, $M_n$ is bipartite with stable vertex sets $V_0$ and $V_1$ constituted by the vertices whose corresponding binary string represents an even or odd number, respectively. Finally, the claimed homomorphism in (e) is simply

$$(u_1,u_2,\dots,u_n)\quad\mapsto\quad(\pi(u_1),\pi(u_2),\dots,\pi(u_n)),$$

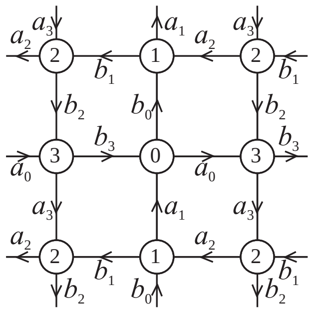


FIG. 2.2. *An alternative definition of the local structure of a 2-dim Manhattan network seen as a 4-partite digraph. (All vertices in $V_j$ are denoted by $j$.)*

where the parity function $\pi$ takes the expected values in $\{0, 1\}$. □

Concerning property ($d$), let us mention that, in [19], the local structure of an standard (2-dim) Manhattan network was introduced as a type of 4-partite digraph in the following way: Let a digraph $G = (V, A)$ have order $N = |V|$ a multiple of 4, with $V = V_0 \cup V_1 \cup V_2 \cup V_3$, where

$$V_j = \{i : 0 \le i \le N - 1; i \equiv j \bmod 4\} \quad (0 \le j \le 3), \tag{2.3}$$

with every vertex $i$ being adjacent to the vertices $i + a_j$, $i + b_j \bmod N$, for some given integers $a_j \equiv 3$, and $b_j \equiv 1 \bmod 4$ satisfying

$$a_0 + a_2 \equiv -a_1 - a_3 \equiv b_0 + b_2 \equiv -b_1 - b_3 \pmod N.$$

See Fig. 2.2 to check that the above conditions impose a Manhattan local structure.

**2.1. The line digraph structure.** Here we show that the standard Manhattan network, which is the two-dimensional case $M_2$, has the structure of a line digraph. To the knowledge of the authors, this relevant fact had not been noticed before. As a consequence, $M_2$ can be seen as the line digraph of a digraph $M_2'$ whose order is one half of the order of $M_2$ and, what is more important, some properties of $M_2$ can be derived from those of $M_2'$.

First, recall that, given a digraph $G = (V, A)$ with $n$ vertices and $m$ arcs, its line digraph $LG = (V_L, A_L)$ has vertices representing the arcs of $G$; so that we identify each vertex $ij \in V_L$ with the arc $(i, j) \in A$; and its adjacencies are naturally induced by the arc adjacencies in $G$. More precisely, vertex $ij \in V_L$ is adjacent to vertex $jk$ since the arc $(i, j) \in A$ has the same terminal vertex as the initial vertex of $(j, k)$. Thus, the order of $LG$ equals the size $m$ of $G$ and, if $G$ is $\delta$-regular, so is $LG$ and it has $\delta n$ arcs. Also, it is known that if $G$ is different from a (directed) cycle and has diameter $D$, then its line digraph $LG$ has diameter $D + 1$; see [10].

LEMMA 2.3. *For any $N_1, N_2$, the 2-dimensional Manhattan network $M_2$ is a line digraph.*

*Proof.* It suffices to check *Heuchenne's condition* [15], which says that a digraph is a line digraph if and only if it has no multiple arc and the out-neighbor (in-neighbor) sets of any two of its vertices are either identical or disjoint. With this aim, assume that two different vertices, $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$, have a common out-neighbor $\boldsymbol{w}$. Then we claim that the arcs $\boldsymbol{u} \to \boldsymbol{w}$ and $\boldsymbol{v} \to \boldsymbol{w}$ must be of different type; that is, one 1-arc and the other a 2-arc. Otherwise, if both were 1-arcs, say, we would have

$$\boldsymbol{w} = (u_1 + (-1)^{u_2}, u_2) = (v_1 + (-1)^{v_2}, v_2),$$

which leads to $u_2 = v_2$ and $u_1 = v_1$, so that $\boldsymbol{u} = \boldsymbol{v}$ against the hypothesis. The same contradiction is reached if we suppose that both adjacencies are 2-arcs. Thus assume, without lose of generality, that $\boldsymbol{u} \to \boldsymbol{w}$ is a 1-arc and $\boldsymbol{v} \to \boldsymbol{w}$ is a 2-arc. Then,

$$\boldsymbol{w} = (u_1 + (-1)^{u_2}, u_2) = (v_1, v_2 + (-1)^{v_1}),$$

whence

$$u_1 = v_1 - (-1)^{u_2} = v_1 - (-1)^{v_2+(-1)^{v_1}} = v_1 + (-1)^{v_2},$$
$$v_2 = u_2 - (-1)^{v_1} = u_2 - (-1)^{u_1+(-1)^{u_2}} = u_2 + (-1)^{u_1},$$

which imply the existence of another common out-neighbor $\boldsymbol{w}'$ such that $\boldsymbol{u} \to \boldsymbol{w}'$ is a 2-arc and $\boldsymbol{v} \to \boldsymbol{w}'$ is a 1-arc:

$$\boldsymbol{w}' = (u_1, u_2 + (-1)^{u_1}) = (v_1 + (-1)^{v_2}, v_2),$$

and we get the claimed result. □

Summarizing, we have seen that two different vertices $\boldsymbol{u}, \boldsymbol{v}$, have the same out-neighborhood if and only if they are of the form

$$\boldsymbol{u} = (a, b), \qquad \boldsymbol{v} = (a + (-1)^b, b + (-1)^a),$$

for some integers $a \in \mathbb{Z}_{N_1}$, $b \in \mathbb{Z}_{N_2}$. Then, according to the parity (equal "●" or distinct "■") of $a, b$, we have the two possible situations shown in Fig. 2.3, on the left. From this, notice that the digraph $G$ where $M_2$ comes from (that is, $M_2 = LG$) is also bipartite, with independent sets {"●"} and {"■"}. In fact, the infinite pattern corresponds to another planar crystallographic group; namely, the one denoted by $p4$.

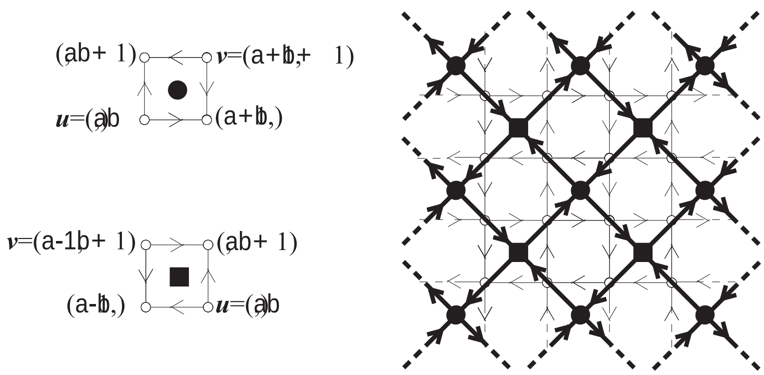

FIG. 2.3. *The local structure of a 2-dim Manhattan network (slim lines and white vertices) and the digraph (thick lines and black vertices) where it comes from as a line digraph.*

It is worthy noting that the property of being line digraphs is not shared, in general, by the Manhattan networks with dimension greater than two. For instance, $M(8, 6, 10)$ does not satisfy Heuchenne's condition since the outneighborhoods $\Gamma^+(1, 1, 5) = \{(2, 5, 5), (7, 2, 5), (7, 5, 6)\}$ and $\Gamma^+(6, 1, 4) = \{(2, 5, 5), (2, 2, 6), (7, 5, 6)\}$ are neither equal nor disjoint.

Two simple consequences of Lemma 2.3 are the following: First, $M_2$ is Hamiltonian, as it is the line digraph of a 2-regular digraph which is Eulerian [5]. In fact, in Section 4 we show that the Hamiltonian property is shared by all $n$-dimensional Manhattan networks; Second, the results in [11] imply that the spectrum of $M_2 = M(N_1, N_2)$ has the eigenvalue 0 with (geometric) multiplicity at least $N_1 N_2 / 2$.

**3. The Colored Automorphism Group.** Here we investigate the symmetries of the Manhattan networks.

THEOREM 3.1. *The n-dim Manhattan network $M_n$ is the Cayley digraph of the group $\Gamma$ with presentation*

$$\Gamma = \langle a_1, a_2, \ldots, a_n \mid a_i^{N_i} = (a_i a_j)^2 = (a_i a_j^{-1})^2 = 1,\ i,j = 1,\ldots,n\rangle. \tag{3.1}$$

*Proof.* Let us show that the mappings $\phi_j$, $1 \le j \le n$, defined by

$$\phi_j(u_1,\ldots,u_j,\ldots,u_n) = (-u_1,\ldots,-u_{j-1},u_j+1,-u_{j+1},\ldots,-u_n). \tag{3.2}$$

are all isomorphisms of $M_n$ mapping $i$-arcs into $i$-arcs. Indeed, let $\gamma_i^+\boldsymbol{u}$ denote the vertex adjacent from vertex $\boldsymbol{u} = (u_1,\ldots,u_n)$ through the $i$-arc. Then, assuming first that $j \ne i$, say $j < i$,

$$\begin{aligned}
\phi_j(\gamma_i^+\boldsymbol{u}) &= \phi_j(u_1,\ldots,u_j,\ldots,u_i+(-1)^{\sum_{k\ne i}u_k},\ldots,u_n)\\
&= (-u_1,\ldots,u_j+1,\ldots,-u_i+(-1)^{1+\sum_{k\ne i}u_k},\ldots,-u_n)\\
&= \gamma_i^+(-u_1,\ldots,u_j+1,\ldots,-u_i,\ldots,-u_n)\\
&= \gamma_i^+\phi_j(\boldsymbol{u}).
\end{aligned}$$

Otherwise, if $j = i$, we have:

$$\begin{aligned}
\phi_i(\gamma_i^+\boldsymbol{u}) &= \phi_i(u_1,\ldots,u_i+(-1)^{\sum_{k\ne i}u_k},\ldots,u_n)\\
&= (-u_1,\ldots,u_i+1+(-1)^{\sum_{k\ne i}u_k},\ldots,-u_n)\\
&= \gamma_i^+(-u_1,\ldots,u_i+1,\ldots,-u_n)\\
&= \gamma_i^+\phi_i(\boldsymbol{u}).
\end{aligned}$$

Therefore, the mappings $\phi_j$, $1 \le j \le n$, are all color-preserving automorphisms of $M_n$. Let us now show that the permutation group $\langle \phi_i \mid 1 \le i \le n\rangle$ acts transitively on the set $\mathbb{Z}\times\mathbb{Z}\times \overset{n}{\cdots} \times\mathbb{Z}$ (and hence, also on the vertex set of $M_n = M(N_1,N_2,\ldots,N_n)$). With this aim, it is enough to show that any vertex $\boldsymbol{u} = (u_1,u_2,\ldots,u_n)$ can be mapped into vertex $\mathbf{0} = (0,0,\ldots,0)$. Let us first distinguish two cases depending on the sign of $u_n$ (the supraindexes of the isomorphisms indicate how many times they are applied):

- $u_n < 0$:

$$(u_1,u_2,\ldots,u_n) \xrightarrow{\phi_n^{|u_n|}} (\pm u_1,\pm u_2,\ldots,0);$$

- $u_n > 0$:

$$\begin{aligned}
(u_1,\ldots,u_i,\ldots,u_n) &\xrightarrow{\phi_i} (-u_1,\ldots,u_i+1,\ldots,-u_n)\\
&\xrightarrow{\phi_n^{u_n}} (\pm u_1,\ldots,\pm(u_i+1),\ldots,0),
\end{aligned}$$

where $i < n$ and, in both cases, the sign in $\pm$ depends on the parity of $u_n$.

Then, by applying the same procedure $n-1$ times, we obtain a vertex of the form $(v_1,0,\ldots,0)$. From this vertex, the desired path is obtained by taking into consideration the following cases. Let $k$ be a nonnegative integer:

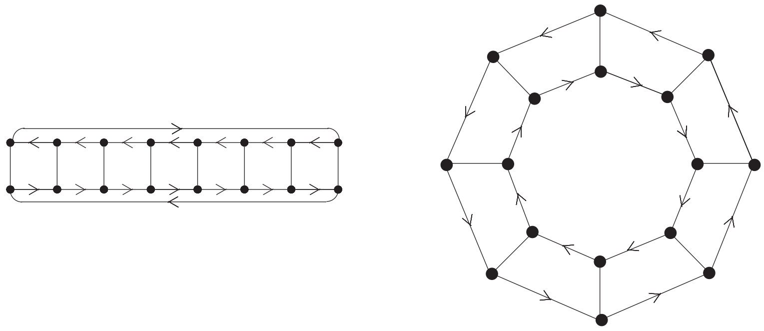

FIG. 3.1. *Standard representations of the Manhattan network $M(8,2)$ and the Cayley digraph of $D_8$ (each line stands for two opposite arcs).*

- $v_1 = -k$:

$$(-k,0,\dots) \xrightarrow{\phi_1^k} (0,0,\dots).$$

- $v_1 = 2k+1$:

$$(2k+1,0,\dots) \xrightarrow{\phi_2} (-2k-1,1,\dots) \xrightarrow{\phi_1^{2k+1}} (0,-1,\dots) \xrightarrow{\phi_2} (0,0,\dots).$$

- $v_1 = 2k$:

$$\begin{aligned}(2k,0,\dots) &\xrightarrow{\phi_2} (-2k,1,\dots) \xrightarrow{\phi_1^{2k}} (0,1,\dots) \xrightarrow{\phi_1} (1,-1,\dots)\\ &\xrightarrow{\phi_2} (-1,0,\dots) \xrightarrow{\phi_1} (0,0,\dots).\end{aligned}$$

Thus, the group $\Gamma = \langle \phi_1,\dots,\phi_n\rangle$ is a regular subgroup of the automorphism group $\mathcal{A}ut\, M_n$ and $M_n$ is a Cayley digraph. Concerning the structure of $\Gamma$, let us check only the second defining relation in (3.1), as the others are proved similarly.

$$\begin{aligned}(\phi_i\phi_j)^2(\boldsymbol{u}) &= \phi_i\phi_j\phi_i\phi_j(u_1,\dots,u_i,\dots,u_j,\dots,u_n)\\ &= \phi_i\phi_j\phi_i(-u_1,\dots,-u_i,\dots,u_j+1,\dots,-u_n)\\ &= \phi_i\phi_j(u_1,\dots,-u_i+1,\dots,-u_j-1,\dots,u_n)\\ &= \phi_i(-u_1,\dots,u_i-1,\dots,-u_j,\dots,-u_n)\\ &= (u_1,\dots,u_i,\dots,u_j,\dots,u_n) = \boldsymbol{u}.\end{aligned}$$

□

This structural result has some appealing consequences, the most evident being the following corollary.

COROLLARY 3.2. *The $n$-dim Manhattan network $M_n$ is a vertex-symmetric* (*but not necessarily arc-symmetric*) *digraph.*

The reader familiar with group theory will have already noted that, in the two-dimensional case, the presentation in (3.1) without the first generating relations $a_1^{N_1} = a_2^{N_2} = 1$ corresponds to the (plane) crystallographic group *pgg* [9]. Consequently, we have the following result:

COROLLARY 3.3. *The underlying Cayley digraph of the* 2*-dim Manhattan network $M_2$, with respect to the arc-coloring defined in* (*2.1*) *is a* (*normal*) *subgroup of the crystallographic group pgg.*

In particular, for $N_1 = n$, $N_2 = 2$, we get the dihedral group $D_n$ (the symmetry group in 2D of a $n$-side regular polygon). See Fig. 3.1 for the standard drawing of $M(8,2)$ and the Cayley digraph of $D_8$.

**3.1. An alternative definition.** The above results imply an alternative presentation of the Manhattan networks.

DEFINITION 3.4. *The vertex set of* $M_n = M(N_1, N_2, \ldots, N_n)$ *is, as above,* $\mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_n}$ *and the (i-)arcs are now:*

$$(u_1, \ldots, u_i, \ldots, u_n) \quad \rightarrow \quad (-u_1, \ldots, -u_{i-1}, u_i + 1, -u_{i+1}, \ldots, -u_n) \qquad (1 \le i \le n). \tag{3.3}$$

LEMMA 3.5. *The graphs defined by (2.1) and (3.3) are isomorphic.*

*Proof.* We introduce the isomorphism from the standard definition to the new presentation as follows ($1 \le i, j \le n$):

$$\Psi(u_1, \ldots, u_i, \ldots, u_n) = ((-1)^{\sum_{j\neq 1} u_j} u_1, \ldots, (-1)^{\sum_{j\neq i} u_j} u_i, \ldots, (-1)^{\sum_{j\neq n} u_j} u_n). \tag{3.4}$$

This application preserves the adjacencies and their "colors". Indeed,

$$\Psi(u_1, \ldots, u_i, \ldots, u_n) = ((-1)^{\sum_{j\neq 1} u_j} u_1, \ldots, (-1)^{\sum_{j\neq i} u_j} u_i, \ldots, (-1)^{\sum_{j\neq n} u_j} u_n) \quad \rightarrow$$
$$(-(-1)^{\sum_{j\neq 1} u_j} u_1, \ldots, (-1)^{\sum_{j\neq i} u_j} u_i + 1, \ldots, -(-1)^{\sum_{j\neq n} u_j} u_n) =$$
$$((-1)^{\sum_{j\neq 1} u_j + (-1)^{\sum_{j\neq i} u_j}} u_1, \ldots, (-1)^{\sum_{j\neq i} u_j} (u_i + (-1)^{\sum_{j\neq i} u_j}), \ldots,$$
$$(-1)^{\sum_{j\neq n} u_j + (-1)^{\sum_{j\neq i} u_j}} u_n) = \Psi(u_1, \ldots, u_i + (-1)^{\sum_{j\neq i} u_j}, \ldots, u_n).$$

□

An an example, Fig. 3.2 shows both, the standard definition and the new presentation of $M(6,4)$. (The torus surface is drawn as usual, where the directed dashed lines represent the identification of parallel sides of the rectangle.)

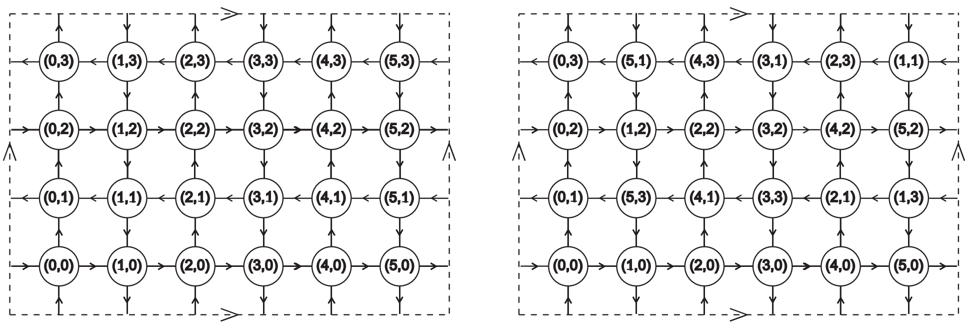


FIG. 3.2. *The vertices with their standard labels and with the labels induced by the applications* $\phi_j$ *in a Manhattan network* $M(6,4)$.

As suggested by this example, it can be readily checked that $\Psi$ is involutive, and hence the mapping from the alternative definition to the standard one is simply $\Psi^{-1} = \Psi$.

**3.2. The Metric Parameters.** In the 2-dimensional case, the diameter of the Manhattan network $M_2$ was first explicitly given in [6] by using spanning trees, although it follows easily from the results in [22] where the mean distance was computed.

The same result was proved in [8] from the comparison of the distance distribution in $M_2$ and the corresponding undirected toroidal mesh. In the last two papers, the distribution of vertices at each distance was also given, which allows closed formulas for the mean distance. In particular, for large values of $N_1$ and $N_2$, the number of vertices at distance $k \geq 4$ from a given vertex, say $\mathbf{0}$, of $M_2(N_1, N_2)$, is $4(k-1)$ (see Fig. 3.3 for the cases $k = 7, 8$). This was also noted in [19] for the (not necessarily toroidal) 2-dim Manhattan network with vertex set as in (2.3). Besides, in the same paper it was shown that, considering the digraph as bipartite, if it has diameter $D(> 4)$, then its order is upper bounded for the following Moore-like bound (see [18]):

$$N(2, D) = \begin{cases} 2(D-1)^2, & D \text{ odd}, \\ 2[(D-1)^2 + 1], & D \text{ even}. \end{cases}$$

In fact, if we do not impose the "toroidal closure" of the network, the above values can be attained since the corresponding tiles (that is, the sets of unit squares associated to the vertices which are at distance $\leq D$ from $\mathbf{0}$) tessellates periodically the plane (see Fig. 3.3). More precisely, when $D$ is odd, the "steps" for attaining the maximum order are, for instance,

$$\begin{array}{llll} a_0 = 3, & a_1 = 2D - 3, & a_2 = -2D + 1, & a_3 = -1; \\ b_0 = 1, & b_1 = -3, & a_2 = -2D + 3, & a_3 = 2D - 1; \end{array}$$

whereas, for $D$ even, we have:

$$\begin{array}{llll} a_0 = -3, & a_1 = 2D + 1, & a_2 = -2D + 1, & a_3 = 1; \\ b_0 = -1, & b_1 = 3, & a_2 = -2D - 1, & a_3 = 2D - 1. \end{array}$$

As the reader will have already noted, in the toroidal case studied here, and for a given number of vertices $N = N_1 N_2$, the diameter is much greater than the obtained above. In this case we have the following known result, which we use afterwards to study the $n$-dim case:

Theorem 3.6. *The diameter of the Manhattan network $M(N_1, N_2)$ is*

*(a)* $D = \frac{N_1}{2} + \frac{N_2}{2} + 1$, *if* $N_1 \equiv N_2 \equiv 0 \pmod 4$*;*

*(b)* $D = \frac{N_1}{2} + \frac{N_2}{2}$, *otherwise.*

*Proof.* For completeness, we here give a proof based on our alternative presentation in Definition 3.4. Let $\alpha, \beta$ be some even and odd integers, respectively, in $[0, N_i/2 - 1]$, $i = 1, 2$. Similarly, let $\gamma$ be any integer in the interval $[0, N_i/2 - 1]$. Because of the symmetry of the digraph, it suffices to consider a path from a generic vertex $\boldsymbol{u} = (u_1, u_2)$ to vertex $\mathbf{0} = (0, 0)$, whose length never exceeds the above values for $D$. With this aim, we first consider the cases where some $u_i$ equals $\pm N_i/2$ (the sign is irrelevant since we are in $\mathbb{Z}_{N_i}$ with $N_i$ even). Here the arrows and the numbers above them represent the followed paths and their lengths.

*(i)* $u_1 = N_1/2$, $u_2 = N_2/2$:

$$(N_1/2, N_2/2) \stackrel{+N_1/2}{\longrightarrow} (0, N_2/2) \stackrel{+N_2/2}{\longrightarrow} (0, 0).$$

*(ii)* $u_1 = N_1/2$ (even), $u_2 = -\gamma$:

$$(N_1/2, -\gamma) \stackrel{+N_1/2}{\longrightarrow} (0, -\gamma) \stackrel{+\gamma}{\longrightarrow} (0, 0).$$

(*iii*) $u_1 = N_1/2$ (even), $u_2 = \alpha$:

$$(N_1/2, \alpha) \xrightarrow{+1} (N_1/2+1, -\alpha) \xrightarrow{+\alpha} (N_1/2+1, 0) \xrightarrow{+N_1/2-1} (0,0).$$

(*iv*) $u_1 = N_1/2$ (even), $u_2 = \beta$:

$$(N_1/2, \beta) \xrightarrow{+N_1/2} (0, \beta) \xrightarrow{+1} (1, -\beta) \xrightarrow{+\beta} (-1, 0) \xrightarrow{+1} (0,0).$$

(*v*) $u_1 = N_1/2$ (odd), $u_2 = \gamma$:

$$(N_1/2, \gamma) \xrightarrow{+N_1/2} (0, -\gamma) \xrightarrow{+\gamma} (0,0).$$

(*vi*) $u_1 = N_1/2$ (odd), $u_2 = -\gamma$:

$$(N_1/2, -\gamma) \xrightarrow{+\gamma} (N_1/2, 0) \xrightarrow{+N_1/2} (0,0).$$

Note that all the above paths have length $\mathrm{dist}(\boldsymbol{u}, \mathbf{0}) \leq N_1/2 + N_2/2$, except in case (*iv*) where we have

$$\mathrm{dist}(\boldsymbol{u}, \mathbf{0}) = \frac{N_1}{2} + \beta + 2 \leq \frac{N_1}{2} + \frac{N_2}{2} + 1, \tag{3.5}$$

and equality is attained when $\beta = N_2/2 - 1$ or, in the symmetric case, $u_1 = \beta = N_1/2 - 1$ and $u_2 = N_2/2$. Note that, in both cases, $N_1/2$ and $N_2/2$ must be even, so that we are in case ($a$) of the theorem. Besides, when $\beta = N_2/2 - 2$, Eq. (3.5) gives $\mathrm{dist}(\boldsymbol{u}, \mathbf{0}) = N_1/2 + N_2/2$ (for $N_1/2$ even and $N_2/2$ odd); and the same holds in the symmetric case $u_1 = \beta = N_1/2 - 2$, $u_2 = N_2/2$ (for $N_1/2$ odd and $N_2/2$ even). Such cases correspond to case (b) in the statement of the theorem.

Moreover, when neither of the entries $u_i$ equals $N_i/2$, we need to consider the following cases:

(1) $u_1 = -\alpha_1$, $u_2 = -\alpha_2$:

$$(-\alpha_1, -\alpha_2) \xrightarrow{+\alpha_1} (0, -\alpha_2) \xrightarrow{+\alpha_2} (0,0).$$

(2) $u_1 = -\alpha$, $u_2 = -\beta$:

$$(-\alpha, -\beta) \xrightarrow{+\alpha} (0, -\beta) \xrightarrow{+\beta} (0,0).$$

(3) $u_1 = -\beta_1$, $u_2 = -\beta_2$:

$$(-\beta_1, -\beta_2) \xrightarrow{+\beta_1} (0, \beta_2) \xrightarrow{+1} (1, -\beta_2) \xrightarrow{+\beta_2} (-1, 0) \xrightarrow{+1} (0,0).$$

(4) $u_1 = -\alpha$, $u_2 = \beta$:

$$(-\alpha, \beta) \xrightarrow{+\alpha} (0, \beta) \xrightarrow{+1} (1, -\beta) \xrightarrow{+\beta} (-1, 0) \xrightarrow{+1} (0,0).$$

(5) $u_1 = -\beta$, $u_2 = \alpha$:

$$(-\beta, \alpha) \xrightarrow{+\beta} (0, -\alpha) \xrightarrow{+\alpha} (0,0).$$

(6) $u_1 = -\alpha_1$, $u_2 = \alpha_2$:

$$(-\alpha_1, \alpha_2) \xrightarrow{+1} (-\alpha_1+1, -\alpha_2) \xrightarrow{+\alpha_2} (-\alpha_1+1, 0) \xrightarrow{+\alpha_1-1} (0,0).$$

(7) $u_1 = -\beta_1$, $u_2 = \beta_2$:

$$(-\beta_1, \beta_2) \xrightarrow{+\beta_1} (0, -\beta_2) \xrightarrow{+\beta_2} (0, 0).$$

(8$a$) $u_1 = \alpha_1 < N_1/2 - 1$, $u_2 = \alpha_2$:

$$\begin{aligned}(\alpha_1, \alpha_2) &\xrightarrow{+1} (\alpha_1 + 1, -\alpha_2) \xrightarrow{+\alpha_2} (\alpha_1 + 1, 0)\\ &\xrightarrow{+1} (-\alpha_1 - 1, 1) \xrightarrow{+\alpha_1+1} (0, -1) \xrightarrow{+1} (0, 0).\end{aligned}$$

(8$b$) $u_1 = \alpha_1 = N_1/2 - 1$, $u_2 = \alpha_2$:

$$(\alpha_1, \alpha_2) \xrightarrow{+1} (\alpha_1 + 1, -\alpha_2) = (N_1/2, -\alpha_2) \xrightarrow{+\alpha_2} (N_1/2, 0) \xrightarrow{+N_1/2} (0, 0).$$

(9) $u_1 = \alpha$, $u_2 = \beta$:

$$(\alpha, \beta) \xrightarrow{+1} (\alpha + 1, -\beta) \xrightarrow{+\beta} (-\alpha - 1, 0) \xrightarrow{\alpha+1} (0, 0).$$

(10) $u_1 = \beta_1$, $u_2 = \beta_2$:

$$(\beta_1, \beta_2) \xrightarrow{+1} (\beta_1 + 1, -\beta_2) \xrightarrow{+\beta_2} (-\beta_1 - 1, 0) \xrightarrow{+\beta_1+1} (0, 0).$$

Observe again that all the above paths have length $\mathrm{dist}(\boldsymbol{u}, \mathbf{0}) < N_1/2 + N_2/2$, except in the cases (4), (8$b$) and (9) where we have

$$\mathrm{dist}(\boldsymbol{u}, \mathbf{0}) = \frac{N_1}{2} + \frac{N_2}{2},$$

for different parities of $N_1/2$ and $N_2/2$. For instance, in case (8$b$) the maximum is attained when $\alpha_2 = N_1/2 - 1$; that is, for a vertex of the form $(N_1/2 - 1, N_2/2 - 1)$ where $N_1 \equiv N_2 \equiv 2 \bmod 4$. (This and all the other vertices at maximum distance are listed below.) This completes the proof. □

Notice that, from the above proof, the vertices at maximum distance from $\mathbf{0}$ are, depending on the case (recall that we are using the alternative definition):

($a$) $N_1 \equiv N_2 \equiv 0 \mod 4$:
($iv$): $(N_1/2, N_2/2 - 1)$, $(N_1/2 - 1, N_2/2)$;

($b1$) $N_1 \equiv 0,\ N_2 \equiv 2 \mod 4$:
($i$): $(N_1/2, N_2/2)$,
($iv$): $(N_1/2, N_2/2 - 2)$,
(4): $(N_1/2 - 1, N_2/2 + 1)$,
(9): $(N_1/2 - 1, N_2/2 - 1)$;

($b2$) $N_1 \equiv 2,\ N_2 \equiv 0 \mod 4$:
($i$): $(N_1/2, N_2/2)$,
($iv$): $(N_1/2 - 2, N_2/2)$,
(4): $(N_1/2 + 1, N_2/2 - 1)$,
(9): $(N_1/2 - 1, N_2/2 - 1)$;

($b3$) $N_1 \equiv N_2 \equiv 2 \mod 4$:
($i$): $(N_1/2, N_2/2)$,
(8$b$): $(N_1/2 - 1, N_2/2 - 1)$.

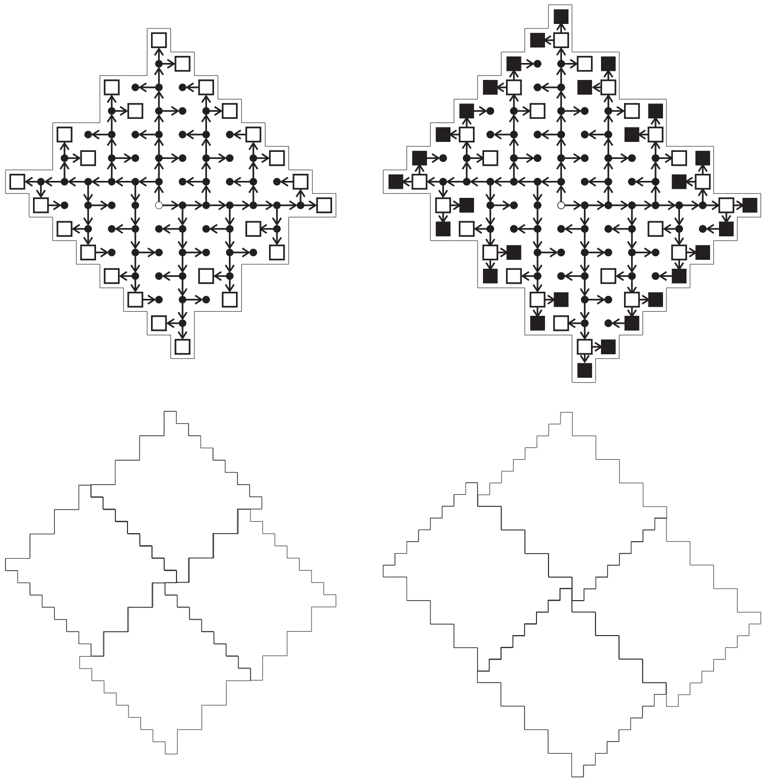

FIG. 3.3. *The vertices at distance* 7 (“□”) *and at distance* 8 (“■”) *from vertex* (0,0) (“$\circ$”), *and the corresponding tessellations.*

To derive the diameter of the $n$-dimensional Manhattan network, it is useful to introduce the following notation. Given $N$ (even) and $0 \le u \le N$, let $\|u\|_N$ be the distance between 0 and $u$ in the undirected cycle $C_N$; that is, $\|u\|_N = \min\{u \,(\mathrm{mod}\, N), -u \,(\mathrm{mod}\, N)\}$ (so that $0 \le \|u\|_N \le N/2$).

LEMMA 3.7. *Let us consider the vertices* $\mathbf{0} = (0, 0, \ldots, 0)$, $\boldsymbol{u} = (u_1, \ldots, u_{n-1}, u_n)$ *in* $M_n = (N_1, \ldots, N_{n-1}, N_n)$, $n > 2$; *and* $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{u}' = (u_1, \ldots, u_{n-1})$ *in* $M_{n-1} = (N_1, \ldots, N_{n-1})$. *Let* $\alpha = (-1)^{u_n}$. *Then,*

($a$) $\mathrm{dist}_{M_n}(\boldsymbol{u}, \mathbf{0}) \ge \sum_{i=1}^{n} \|u_i\|_{N_i}$;
($b$) $\mathrm{dist}_{M_n}(\boldsymbol{u}, \mathbf{0}) \le \mathrm{dist}_{M_{n-1}}(\alpha \boldsymbol{u}', \mathbf{0}) + \|u_n\|_{N_n}$, *if* $u_n \in [N_n/2, N_n]$;
($c$) $\mathrm{dist}_{M_n}(\boldsymbol{u}, \mathbf{0}) \le \mathrm{dist}_{M_{n-1}}(\alpha(\boldsymbol{u}' + \boldsymbol{e}_1), \mathbf{0}) + \|u_n\|_{N_n} + 1$, *if* $u_n \in [0, N_n/2 - 1]$;

*Proof.* The inequality in ($a$) is a direct consequence of the fact that the underlying graph of $M_n$ is just the direct product of the cycles $C_{N_i}$, $1 \le i \le N_i$.

If $N_n/2 \le u_n \le N_n$, there is a path of length $N_n - u_n = \|u_n\|_{N_n}$ from $\boldsymbol{u}$ to $(\alpha u_1, \ldots, \alpha u_{n-1}, 0)$ in $M_n$. From this vertex, we will need, at most, $\mathrm{dist}_{M_{n-1}}(\alpha \boldsymbol{u}', \mathbf{0})$ steps to reach $\mathbf{0}$. This proves ($b$).

Otherwise, when $0 \le u_n \le N_n/2 - 1$, we go first, in one step, from $\boldsymbol{u}$ to $(u_1 + 1, -u_2, \ldots, -u_{n-1}, -u_n)$. Then, from this vertex we have a path of length $u_n = \|u_n\|_{N_n}$ to $(\alpha(u_1 + 1), -\alpha u_2, \ldots, -\alpha u_{n-1}, 0)$, and reasoning as above we get ($c$).
□

As a consequence, from the vertex symmetry of the involved digraphs, we have:

$$\sum_{i=1}^{n} \frac{N_i}{2} \leq \mathrm{ecc}_{M_n}(\mathbf{0}) \leq \mathrm{ecc}_{M_{n-1}}(\mathbf{0}) + \frac{N_n}{2}$$

since, in case $(c)$, $\|u_n\|_{N_n} \leq N_n/2-1$, and the same formula applies for the respective diameters:

$$\sum_{i=1}^{n} \frac{N_i}{2} \leq D(M_n) \leq D(M_{n-1}) + \frac{N_n}{2}. \tag{3.6}$$

THEOREM 3.8. *The diameter of the $n$-dimensional Manhattan network $M_n = M(N_1, \ldots, N_n)$ is*

(*a*) $D(M_n) = \displaystyle\sum_{i=1}^{n} \frac{N_i}{2} + 1$, *if* $N_i \equiv 0 \pmod 4$ *for any* $1 \leq i \leq n$;

(*b*) $D(M_n) = \displaystyle\sum_{i=1}^{n} \frac{N_i}{2}$, *otherwise.*

*Proof.* Since Lemma 3.7 clearly applies for any two components of the vertex $(u_1, u_2, \ldots, u_n)$, we can apply recursively the above results to get

$$D(M_n) \leq D(M(N_j, N_k)) + \frac{1}{2} \sum_{i \neq j,k} N_i \qquad (1 \leq j < k \leq n). \tag{3.7}$$

Then, under the hypothesis in $(b)$, there is some $1 \leq j \leq n$ such that $N_j \not\equiv 0 \pmod 4$ and $D(M(N_j, N_k)) = N_j/2 + N_k/2$ by Theorem 3.6. This, together with the lower bound in (3.6), proves the equality in $(a)$.

Otherwise, if $N_i \equiv 0 \pmod 4$ for any $1 \leq i \leq n$, Theorem 3.6 and (3.7) yield

$$D(M_n) \leq \frac{1}{2} \sum_{i=1}^{n} N_i + 1. \tag{3.8}$$

Thus, to prove $(a)$ we only need to show that equality is attained for some vertex. In fact, the vertices at maximum distance to $\mathbf{0}$ are, in this case:

$$\boldsymbol{u}_i = (N_1/2, N_2/2, \ldots, N_i/2 - 1, \ldots, N_n/2), \quad (1 \leq i \leq n)$$

(compare with the 2-dim case $(a)$, after the proof of Theorem 3.6). For instance, let us check that $\mathrm{dist}(\boldsymbol{u}_1, \mathbf{0})$ attains the upper bound in (3.8). With this aim, note that the first entry of $\boldsymbol{u}_1$, $N_1/2 - 1$, is odd, whereas the others, $N_i/2$, $2 \leq i \leq n$, are even. Moreover, for all $1 \leq i \leq n$, we need at least $N_i/2$ steps to bring the $i$-th entry to 0, which gives $\mathrm{dist}(\boldsymbol{u}_1, \mathbf{0}) \geq \sum_{i=1}^{n} N_i/2$. In particular, this requieres to change, in some step, the first component from $N_1/2 - 1$ to $-N_1/2 + 1 = N_1/2 + 1$, which is acomplished when the number, say $r$, of previous steps going through $j$-arcs,

$$(u_1, \ldots, u_j, \ldots, u_n) \quad \rightarrow \quad (-u_1, \ldots, u_j + 1, \ldots, -u_n) \quad (2 \leq j \leq n)$$

is odd. Then, although we have given $r$ steps in the "right direction" to $\mathbf{0}$, there is some $j$ such that the first and $j$-th entries of the reached vertex $\boldsymbol{u}'$, $u'_1 = N_1/2 + 1$ and $u'_j \in [N_j/2 + 1, N_j - 1]$, are odd. But now, as in case (3) in the proof of Theorem 3.6, it is impossible to bring these components to 0 without spending at least $\|u'_1\| + \|u'_j\| + 2$ steps (no matter what the other entries are). Hences, it must be $\mathrm{dist}(\boldsymbol{u}_1, \mathbf{0}) = \sum_{i=1}^{n} N_i/2 + 1$ and this completes the proof. □

**4. Hamiltonian Cycles.** In this section we first show that the Manhattan networks are Hamiltonian digraphs. Moreover, in the 2-dim case, some necessary and sufficient conditions for $M_2$ to be decomposable into two (arc-disjoint) Hamiltonian cycles are derived.

THEOREM 4.1. *The Manhattan network $M_n$ is Hamiltonian.*

*Proof.* We proceed by induction of $n$. For $n = 1$, $M_1$ can be seen as a directed cycle and hence it is trivially Hamiltonian. (We can also start from $N = 2$ since we already know that $M_2$ is Hamiltonian by Lemma 2.3.) Now suppose that there exists a Hamiltonian cycle for $M_{n-1}$. Then, we build a Hamiltonian cycle for $M_n$ by appropriately joining $N_n$ Hamiltonian cycles (without some arcs) of its $N_n$ subdigraphs isomorphic to $M_{n-1}$ (remember Lemma 2.2(c)). More precisely, the cycle begins, say, in $(u_1, u_2, \ldots, u_{n-2}, 0, 0)$ and goes from this vertex to the vertices $(u_1, u_2, \ldots, u_{n-2}, 0, 1), \ldots, (u_1, u_2, \ldots, u_{n-2}, 0, N_n - 1)$. From this last vertex, we follows a clockwise Hamiltonian cycle in a subdigraph $M_{n-1}$ (without the last step) until vertex $(u_1, u_2, \ldots, u_{n-2}, N_{n-1} - 1, N_n - 1)$. Then, we go to vertex $(u_1, u_2, \ldots, u_{n-2}, N_{n-1} - 1, N_n - 2)$ and, from it, we follows counterclockwise cycle until $(u_1, u_2, \ldots, u_{n-2}, 1, N_n - 2)$. From here, we go to $(u_1, u_2, \ldots, u_{n-2}, 1, N_n - 3)$. Now we repeat this patters several times until we reach the counterclockwise Hamiltonian cycle of a $M_{n-1}$ (without the last step), which finishes in $(u_1, u_2, \ldots, u_{n-2}, 0, 0)$ and this closes the Hamiltonian cycle of $M_n$ (see Figure 4.1). □

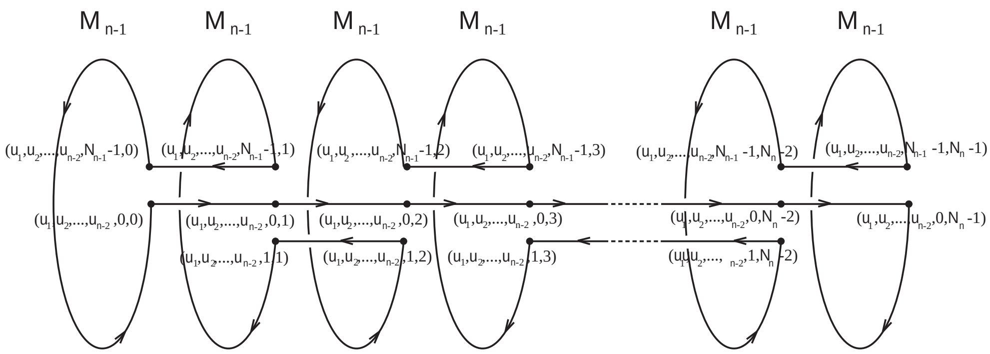


FIG. 4.1. *A Hamiltonian cycle in $M_n$.*

As an example of a Hamiltonian cycle in $M(N_1, N_2)$, see Figure 4.2.

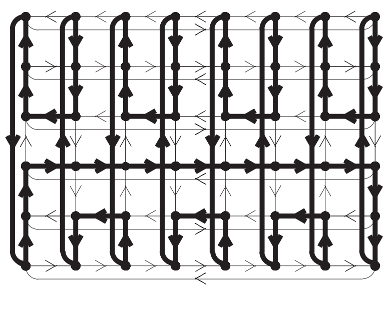

FIG. 4.2. *A Hamiltonian cycle in $M(8, 6)$.*

Besides the Hamiltonian property, in some applications it is interesting to have a decomposition of the network into (two or more) edge-disjoint Hamiltonia cycles.

In this context, Varvarigos [22] showed the presence of two edge-disjoint Hamiltonian cycles in the 2-dim Manhattan network $M(N,N)$. Generalizing his result, we give here some sufficient conditions for $M(N_1,N_2)$ to be Hamiltonian decomposable in such a way.

PROPOSITION 4.2. *Let $M_2 = M(N_1,N_2)$ be a 2-dim Manhattan network. If the two following conditions hold:*

(*a*) $\gcd\{N_1/2, N_2/2\} = d \geq 2$*;*

(*b*) *There exist $d_1, d_2 > 0$, $d_1 + d_2 = d$, such that*

$$\gcd\{d_1, N_1/2\} = \gcd\{d_2, N_2/2\} = 1;$$

*then $M_2$ contains two edge-disjoint Hamiltonian cycles.*

*Proof.* By the results in [21] (see also [12]), the above conditions imply that the cartesian product of the directed cycles $C_{N_1/2} \times C_{N_2/2}$ is Hamiltonian. Then, Fig. 4.3 illustrates the way how such a Hamiltonian cycle (with arcs going east and north) induces a Hamiltonian cycle in $M_2$. The key idea is that, before closing, this east-north cycle changes its directions to go west and south. Then, at the end, it changes again to connect with the first subpath and completing the Hamiltonian cycle in $M_2$. With respect to the other (edge-disjoint) cycle of $M(N_1,N_2)$, it is simply the complement of the first one. (Or, alternatively, it can be seen as a Hamiltonian cycle constructed in the same way as the first one, but on the converse digraph of the Manhattan network $M(N_2,N_1)$; see again the figure for the details.) □

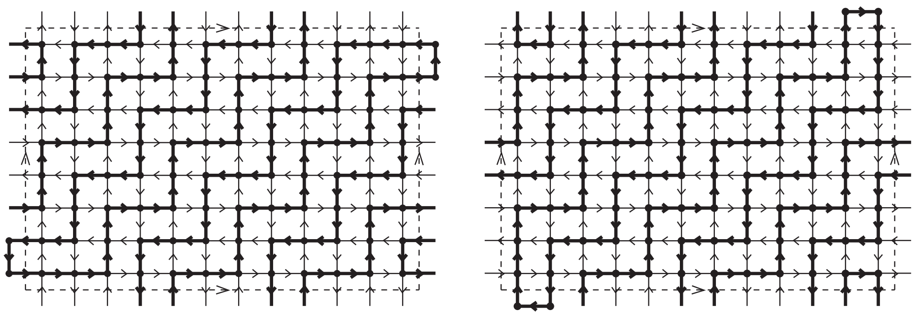

FIG. 4.3. *A decomposition of $M(12,8)$ into two edge-disjoint Hamiltonian cycles.*

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